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REDUCTION CRITERIA FOR MODULES

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Section 1. Introduction.

Let (R, m) be a d-dimensional quasi-unmixed local ring and $J \subseteq I$ m-primary ideals. A theorem of fundamental importance, due to D. Rees, states that the multiplicity of J equals the multiplicity of I if and only if J is a reduction of I (see [R1]). This result was extended to equimultiple ideals by Böger, and further, by Rees himself, to ideals $J \subseteq I$ whose quotient has finite length (see [B] and [R2]). Recently, in [KR] Kirby and Rees and in [KT] Kleiman and Thorup have extended the theorem to a more general situation wherein the focus is turned to a pair of graded R-algebras rather than a pair of ideals. The methods used in [KR] involve complexes of graded modules and Euler-Poincare characteristics, while the methods in [KT] rely upon intersection theory for abstract Noetherian schemes. The purpose of this note is two-fold. On the one hand, we wish to give a proof of the Rees multiplicity theorem for modules in a relatively straight forward way which appeals only to basic ideas from the theory of Hilbert functions. On the other hand, we would like to demonstrate module forms of reduction criteria obtained by Böger in [B]. In particular, we define the notion of an equimultiple module and provide an extension of Böger's theorem from equimultiple ideals to equimultiple modules (also see [KT; Thm 10.9]).

We begin by establishing our set-up. Fix a d-dimensional local ring (R, m) and let $B \subseteq A \subseteq F := R^N$ be finitely generated R-modules. We assume that A and B have rank r. By this we shall mean that every $r+1 \times r+1$ minor of the matrix associated to A (resp. B) equals zero and that the ideal of $r \times r$ minors has height at least one. We denote this ideal by $I_r(A)$ (resp. $I_r(B)$). Let $\mathcal{B} \subseteq \mathcal{A} \subseteq R[X_1, \ldots, X_N]$ be, respectively, the R-algebras generated by the linear forms in $X_1, \ldots, X_N$ corresponding to the generators of B and A.

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Our goal is to determine when $\mathcal{A}$ is a finitely generated $\mathcal{B}$-module. Equivalently, we seek to determine when the R-algebra $\mathcal{A}$ is integrally dependent over the ring $\mathcal{B}$. When either of these conditions hold, B is said to be a *reduction* of A. For example, if B were the ith module of boundaries and A the ith module of cycles in a complex of free R-modules having finite length homology, then it is an issue of great interest to determine when B is a reduction of A. Indeed, a number of unsettled homological conjectures can be reduced to precisely this question. Hence our interest in the special case at hand.

Section 2. The criteria.

Before presenting our treatment of Rees' theorem we state a crucial lemma.

Lemma 2.1. *Let $\mathcal{R}$ be a finitely generated bigraded algebra, generated in degrees $(1,0)$ and $(0,1)$ over the Artinian local ring (Q,n). Let $\mathcal{M}$ be a finitely generated bigraded $\mathcal{R}$-module. For all $t,s \geq 0$, set $H(t,s) := \lambda_Q(\mathcal{M}_{t,s})$, the length of $\mathcal{M}_{t,s}$. Assume $H(t,s) \neq 0$, for t,s sufficiently large. Then :*

(1) *There exists a polynomial $P(X,Y)$ with rational coefficients such that $H(t,s) = P(t,s)$, for all $t >> 0, s >> 0$. Furthermore, $\deg(P(X,Y)) = \mathrm{rdim}(\mathcal{M}) - 2$.*
(2) *Set $H(n) = \sum_{j=0}^{n} H(n-j,j)$. Then for $n >> 0$, $H(n)$ assumes the values of a polynomial with rational coefficients having degree $1 + \deg(P(X,Y))$.*

Remark. Recall that $\mathrm{rdim}(\mathcal{M})$, the relevant dimension of $\mathcal{M}$, equals the largest value $\dim(\mathcal{R}/\mathcal{P})$ where $\mathcal{P}$ is a relevant prime containing $\mathrm{ann}_{\mathcal{R}}(\mathcal{M})$. A prime ideal $\mathcal{P}$ is relevant if $\mathcal{R}_{t,s} \not\subseteq \mathcal{P}$, for all $(t,s) \neq (0,0)$. The proof of part (2) of the lemma is standard while the first part is more or less well-known. See [KR] or [VKM]. Various forms of the following theorem can be found in [K], [R2], [KR] and [KT]. We present a proof for the sake of completeness and, we hope, clarity.

Theorem 2.2. *Suppose $\lambda_R(A/B) < \infty$. Then :*

(1) *For all $n >> 0$, $\lambda_R(\mathcal{A}_n/\mathcal{B}_n)$ assumes the values of a polynomial $P(n)$ in n having rational coefficients and degree less than or equal to $d + r - 1$.*
(2) *If $\mathcal{A}$ is a finitely generated $\mathcal{B}$-module, $\deg(P(n)) < d + r - 1$.*
(3) *If R is quasi-unmixed, the converse of (2) holds.*

Proof. It follows readily from the assumption $\lambda_R(A/B) < \infty$ that $\lambda_R(\mathcal{A}_n/\mathcal{B}_n) < \infty$, for all $n \geq 1$. Furthermore, it is not hard to show that $\dim(\mathcal{A}) = \dim(\mathcal{B}) = d + r$. Consequently, if $\mathcal{A}$ is a finitely generated $\mathcal{B}$-module, parts (1) and (2) follow from standard Hilbert function theory, since $\dim(\mathcal{B}/\mathrm{ann}_{\mathcal{B}}(\mathcal{A}/\mathcal{B})) \leq d + r - 1$. This latter fact follows because $I_r(B)$ is contained in the nilradical of $\mathrm{ann}_{\mathcal{B}}(\mathcal{A}/\mathcal{B})$. Now suppose that R is quasi-unmixed and $\mathcal{A}$ is *not* a finitely generated $\mathcal{B}$-module. Let U be an additional indeterminate and set

$$\mathcal{R} = \mathcal{A}[b_1 U, \ldots, b_k U]$$

where $b_1, \ldots, b_k$ are the linear forms corresponding to the generators of B. Note that $\mathcal{R} = \bigoplus_{t,s\geq 0} \mathcal{A}_t\mathcal{B}_s U^s$ is a bigraded R-algebra on the one hand and equals the Rees ring of $\mathcal{A}$ with respect to $\mathcal{B}_1\mathcal{A}$ on the other hand. Note also that $\dim(\mathcal{R}) = d+r+1$. Consider the bigraded $\mathcal{R}$-module

$$\mathcal{A}_1\mathcal{R}/\mathcal{B}_1\mathcal{R} = \bigoplus_{t\geq 0, s\geq 0} \mathcal{A}_{t+1}\mathcal{B}_s/\mathcal{A}_t\mathcal{B}_{s+1}.$$

Since $\lambda_R(\mathcal{A}_{t+1}\mathcal{B}_s/\mathcal{A}_t\mathcal{B}_{s+1}) < \infty$, $[\mathcal{A}_1\mathcal{R}/\mathcal{B}_1\mathcal{R}]_{t,s}$ has finite length for all $t \geq 0, s \geq 0$ as a module over the Artinian local ring $[\mathcal{R}/\operatorname{ann}_{\mathcal{R}}(\mathcal{A}_1\mathcal{R}/\mathcal{B}_1\mathcal{R})]_0$. Since

$$\dim(\mathcal{R}/\operatorname{ann}_{\mathcal{R}}(\mathcal{A}_1\mathcal{R}/\mathcal{B}_1\mathcal{R})) \leq d+r,$$

by the lemma there exists a polynomial $Q(X,Y)$ with rational coefficients satisfying

$$\deg(Q(X,Y)) \leq d+r-2 \quad \textit{and} \quad \lambda_R(\mathcal{A}_{t+1}\mathcal{B}_s/\mathcal{A}_t\mathcal{B}_{s+1}) = Q(t,s)$$

for all $t >> 0, s >> 0$. Since

$$\lambda_R(\mathcal{A}_n/\mathcal{B}_n) = \sum_{j=0}^{n-1} \lambda_R(\mathcal{A}_{n-j}\mathcal{B}_j/\mathcal{A}_{n-j-1}\mathcal{B}_{j+1}),$$

the lemma implies that for $n >> 0$, $\lambda_R(\mathcal{A}_n/\mathcal{B}_n)$ is a polynomial in n having degree equal to $1 + \deg(Q(X,Y))$. It therefore remains to see that $\operatorname{rdim}(\mathcal{A}_1\mathcal{R}/\mathcal{B}_1\mathcal{R}) = d+r$. For this, it suffices to see that there exists a relevant height one prime ideal in $\mathcal{R}$ containing $\operatorname{ann}_{\mathcal{R}}(\mathcal{A}_1\mathcal{R}/\mathcal{B}_1\mathcal{R})$ (since R is quasi-unmixed). Now, since $\mathcal{A}$ is not finite over $\mathcal{B}$, $\mathcal{B}_1\mathcal{A}$ and $\mathcal{A}_+$ have different nilradicals. Choose $P \subseteq \mathcal{A}$ minimal over $\mathcal{B}_1\mathcal{A}$, $\mathcal{A}_+ \not\subseteq P$. Let $\mathcal{P} \subseteq \mathcal{R}$ be a minimal prime over $\mathcal{B}_1\mathcal{R}$ with $\mathcal{P} \cap \mathcal{A} = P$. Since $\mathcal{R}$ is a Rees algebra, $\mathcal{B}_1 U \not\subseteq \mathcal{P}$. Thus $\mathcal{P}$ is relevant and has height one. Since $(\mathcal{A}_1\mathcal{R}/\mathcal{B}_1\mathcal{R})_{\mathcal{P}} = \mathcal{R}_{\mathcal{P}}/\mathcal{B}_1\mathcal{R}_{\mathcal{P}}$, $\operatorname{ann}_{\mathcal{R}}(\mathcal{A}_1\mathcal{R}/\mathcal{B}_1\mathcal{R}) \subseteq \mathcal{P}$, and the proof is complete.

Remark (i) The key idea of considering the bigraded filtration

$$\mathcal{A}_n \supseteq \cdots \supseteq \mathcal{A}_{n-j}\mathcal{B}_j \supseteq \cdots \supseteq \mathcal{B}_n$$

can be traced back to the original paper of Rees, [R1].

(ii) Let $M \subseteq F$ be a finitely generated R-module satisfying $\lambda_R(F/M) < \infty$ and $M \neq F$. If we take $B = M$ and $A = F$, then Theorem 2.2 implies that for all $n >> 0$, $\lambda_R(Sym_n(F)/S_n(M))$ assumes the values of a polynomial with rational coefficients, where $Sym_n(F)$ denotes the nth component of the symmetric algebra of F and $S_n(M)$ denotes the image of the nth component of $Sym(M)$ in $Sym_n(F)$. Since M is clearly not a reduction of F, the theorem implies that the degree of this polynomial is $d+N-1$. This result is due to Buchsbaum-Rim (c.f. [BR]). The normalized leading coefficient of this polynomial is called the Buchsbaum-Rim multiplicity of M. We denote this by $e(M)$. If, in our original set-up, we assume $B \subseteq A \subsetneq F$ and that both A and B have finite colength in F, then we conclude from Theorem 2.2 that, if R is quasi-unmixed, then B is a reduction of A if and only if the Buchsbaum-Rim multiplicity of B equals the Buchsbaum-Rim multiplicity of A. This is a module-theoretic analogue of the original Rees multiplicity theorem.

Before continuing, we will recall the notions of integral closure and analytic spread as they apply to modules $B \subseteq F$ (c.f., [R3]). The *integral closure* of B (in F), denoted B^*, is the largest submodule of F having B as a reduction. Alternately, B^* may be described as the degree one component of the integral closure of $\mathcal{B}$ in $R[X_1, \dots, X_N]$. When R is a domain, these definitions agree with the one given by Rees in [R3], where B^* is defined to be the intersection of the modules $BV \cap F$ as V ranges over the discrete valuation domains between R and its quotient field. The *analytic spread* of B, denoted $s(B)$, is the Krull dimension of the ring $\mathcal{B}/m\mathcal{B}$. In the next proposition we extend an important result due to Ratliff from ideals to modules. Ratliff's theorem (c.f., [Rt1]) states that in a quasi-unmixed local ring, any prime ideal associated to the integral closure of an ideal has height less than or equal to the analytic spread of the ideal. Using this fact, Ratliff was able to give a quick proof of Böger's theorem for equimultiple ideals. In fact he went on to show that the Rees-Böger theorems *characterize* quasi-unmixed local rings. See [Rt1].

Proposition 2.3. *Let R be a locally quasi-unmixed Noetherian ring and $B \subseteq F$ a finitely generated R-module having rank r. Let $P \in \operatorname{Ass}(F/B^*)$. Then* $\operatorname{height}(P) \leq s(B_P) - r + 1$.

Proof. First, we may localize and assume that P is the unique maximal ideal. Second, it is not difficult to show that there exists a minimal prime $z \subseteq P$ such that $P/z \in \operatorname{Ass}((F/zF)/(B')^*)$, where B' is the image of B in F/zF. Since R is quasi-unmixed, $\operatorname{height}(P) = \operatorname{height}(P/z)$. Moreover, $s(B') \leq s(B)$. Thus we may replace R by R/z and assume further that R is a domain. Now, write $P = (B^* : v)$, for some $v \in F \backslash B^*$. Then $Pv \subseteq B^*$. Writing $\tilde{v}$ for the linear form in $R[X_1, \dots, X_N]$ corresponding to v, we obtain $P\tilde{v} \subseteq \mathcal{B}^*$, the integral closure of $\mathcal{B}$. Since $v \notin B^*$, $\tilde{v} \notin \mathcal{B}^*$. It follows that there exists a height one prime $\mathcal{P}^*$ in (the Krull domain) $\mathcal{B}^*$ such that $\tilde{v} \notin \mathcal{B}^*_{\mathcal{P}^*}$. Hence $P \subseteq \mathcal{P}^*$. Therefore $\mathcal{P} \cap R = P$, where $\mathcal{P} = \mathcal{P}^* \cap \mathcal{B}$. Since $\mathcal{B}$ is locally quasi-unmixed, $\operatorname{height}(\mathcal{P}) = 1$ (c.f. [Rt2]). By the dimension formula

$$\operatorname{height}(P) + \operatorname{tr.deg.}_R(\mathcal{B}) = 1 + \operatorname{tr.deg.}_{R/P}(\mathcal{B}/\mathcal{P}) \leq 1 + s(B).$$

Since $\operatorname{tr.deg.}_R(\mathcal{B}) = r$, $\operatorname{height}(P) \leq s(B) - r + 1$, as desired.

We now extend [R2 ; Thm. 3.2] to modules. Note that the first part of the "if" direction appears as [R3 ; Thm 2.5], though our proof is a bit different. In the statement of Theorem 2.4, we write $e(A_Q/B_Q)$ to denote the normalized leading coefficient of the degree $\dim(R_Q) - r + 1$ term of the polynomial derived in Theorem 2.2.

Theorem 2.4. *Let R be a quasi-unmixed local ring and $B \subseteq A \subseteq F$ as above. Suppose that* $\text{height}(B : A) \geq s(B) - r + 1$. *Then B is a reduction of A if and only if either* $\text{height}(B : A) > s(B) - r + 1$ *or* $\text{height}(B : A) = s(B) - r + 1$ *and $e(A_Q/B_Q) = 0$ for every minimal prime Q over $(B : A)$.*

Proof. Let $Q' \in \text{Ass}(F/B^*)$. From the previous proposition we obtain

$$\text{height}(Q') \leq s(B_{Q'}) - r + 1 \leq s(B) - r + 1.$$

Thus, if $\text{height}(B : A) > s(B) - r + 1$, $(B : A) \not\subseteq Q'$ and $B_{Q'} = A_{Q'}$. In particular, $A_{Q'} \subseteq B^*_{Q'}$. If $\text{height}(B : A) = s(B) - r + 1$, then $\text{height}(Q') \leq \text{height}(B : A)$. Therefore, either $(B : A) \not\subseteq Q'$ in which case $A_{Q'} \subseteq B^*_{Q'}$, or Q' is minimal over $(B : A)$. In the second case, localizing at Q' and applying Theorem 2.2 gives $A_{Q'} \subseteq B^*_{Q'}$, as desired.

We need a definition before we can state the extension of Böger's theorem (see also [KT; Thm. 10.9]). Recall that an ideal $I \subseteq R$ is said to be *equimultiple* if $s(I) = \text{height}(I)$. For $B \subseteq F$ as above, we say that B is equimultiple if $r := \mathit{rank}(B) = \mathit{rank}(F)$ and $s(B) = \text{height}(I_r(B)) + r - 1$.

Theorem 2.5. *Let R be a quasi-unmixed local ring and $B \subseteq A \subseteq F$ as above. Assume that B is equimultiple and that $I_r(A)$ and $I_r(B)$ have the same nilradical. Then B is a reduction of A if and only if $e(B_Q) = e(A_Q)$ for all primes Q minimal over $I_r(B)$.*

Proof. First note that for any prime Q minimal over $I_r(B)$, $\lambda_R(F_Q/B_Q) < \infty$, so the Buchsbaum-Rim multiplicities $e(B_Q)$ and $e(A_Q)$ make sense. Now, if B is a reduction of A, then part (2) of Theorem 2.2 implies that $e(B_Q) = e(A_Q)$ for all primes Q minimal over $I_r(B_Q)$. For the converse, let $Q' \in \text{Ass}(F/B^*)$. Then $I_r(B) \subseteq Q'$ and Proposition 2.3 implies that

$$\text{height}(Q') \leq s(B_{Q'}) - r + 1 \leq s(B) - r + 1 = \text{height}(I_r(B)) \leq \text{height}(Q').$$

Therefore Q' is minimal over $I_r(B)$. Part (3) of Theorem 2.2 implies that $A_{Q'} \subseteq B^*_{Q'}$. It follows that $A \subseteq B^*$, i.e., B is a reduction of A.

The last theorem is a module form of a second reduction criterion due to Böger (see [B; Satz 2]). The proof, however, is entirely different as it avoids the use of superficial elements.

Theorem 2.6. *Let R be a quasi-unmixed local ring and $B \subseteq A \subseteq F$ as above. Then B is a reduction of A if and only if the image of B in F/PF is a reduction of the image of A for all prime ideals $P \subseteq R$ satisfying* $\dim(R/P) = 1$.

Proof. Since $\mathcal{A}$ is a finite $\mathcal{B}$-module if and only if $\mathit{rad}(\mathcal{B}_1\mathcal{A}) = \mathit{rad}(\mathcal{A}_+)$, it suffices to assume the dimension one condition and show that $\mathit{rad}(\mathcal{B}_1\mathcal{A}) = \mathit{rad}(\mathcal{A}_+)$. For this, by Krull's principal ideal theorem, it suffices to prove the following statement. For all primes $\mathcal{P} \subseteq \mathcal{A}$ with $\dim(\mathcal{A}/\mathcal{P}) = 1$, if $\mathcal{B}_1\mathcal{A} \subseteq \mathcal{P}$, then $\mathcal{A}_+ \subseteq \mathcal{P}$. In other words, if $\dim(\mathcal{A}/\mathcal{P}) = 1$ and $\mathcal{A}_+ \not\subseteq \mathcal{P}$, then $\mathcal{B}_1\mathcal{A} \not\subseteq \mathcal{P}$. For this it suffices to show that there exists a prime $P \subseteq R$ with $\dim(R/P) = 1$ and $P^* \subseteq \mathcal{P}$, where $P^* = PR[X_1, \ldots, X_N] \cap \mathcal{A}$. After all, by hypothesis, $rad((\mathcal{A}_+, P^*)/P^*) = rad((\mathcal{B}_1\mathcal{A}, P^*)/P^*)$.

To see that such a P exists, we first note that if $dim(\mathcal{A}/\mathcal{P}) = 1$ and $\mathcal{A}_+ \not\subseteq \mathcal{P}$, then $\mathcal{P} \cap R = m$. Indeed, $\mathcal{A}/\mathcal{P}$ is a one dimensional graded algebra which is a domain. Since $(\mathcal{A}/\mathcal{P})_+$ is a non-zero prime, it must be the unique homogeneous maximal ideal, so $(\mathcal{A}/\mathcal{P})_0$ is a field. Thus, $\mathcal{P} \cap R = m$. Now, set $I := I_r(A)$. As we may clearly assume that $\dim(R) > 1$, let $\mathcal{Q} \subseteq \mathcal{P}$ satisfy $\dim(\mathcal{A}/\mathcal{Q}) = 2$ and $I\mathcal{A} \not\subseteq \mathcal{Q}$. Then $\text{height}(\mathcal{Q}) = d + r - 2$. Set $P = \mathcal{Q} \cap R$ and select a minimal prime $z \subseteq P$ which satisfies $\text{height}(\mathcal{Q}/z^*) = d + r - 2$. Applying the dimension formula to the extension $R/z \subseteq \mathcal{A}/z^*$, we obtain

$$\text{height}(P/z) + r = d + r - 2 + \text{tr.deg.}_{R/P}(\mathcal{A}/\mathcal{Q}).$$

Therefore $\text{height}(P) \geq d - 2 + \text{tr.deg.}_{R/P}(\mathcal{A}/\mathcal{Q})$. Since $I \not\subseteq P$, $P \neq m$. Moreover, since $\mathcal{A}_+ \not\subseteq \mathcal{Q}$, $\text{tr.deg.}_{R/P}(\mathcal{A}/\mathcal{Q}) \geq 1$. It follows that $\text{tr.deg.}_{R/P}(\mathcal{A}/\mathcal{Q}) = 1$ and $\text{height}(P) = d - 1$. If we now let S denote the complement of P in R, we have $P_S = \mathcal{Q}_S \cap R_S$. But $\mathcal{A}_S$ is a polynomial ring in a subset of the variables $X_1, \ldots, X_N$ (up to a change of coordinates). Thus $\mathcal{Q}_S$ contains $P\mathcal{A}_S$ and therefore $\mathcal{Q}$ contains P^*. This completes the proof.

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